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High Energy Physics Summer School 2011

Pre-school Notes and Problems

Quantum Field Theory

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Introduction to QED and QCD

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1 Quantum Field Theory

The aim of these notes is to review some concepts which will be assumed for the quantum field theory (QFT) course, and also provide some example problems with which you can test your understanding.

We will build on ideas already encountered by the reader, where the assumed structure of this knowledge can be summarised as in figure 1. On the left-hand side, we have classical mechanics, in which there are two types of entity - *particles* and *fields*. The former obey Newton's laws, but can also be described by the Lagrangian or Hamiltonian formalisms. The power of the latter approach is that one can also describe fields using such a framework, as we will see. In the upper right-hand corner, we have the quantum mechanics of particles. As the reader is no doubt aware, there is a well-defined prescription for taking any classical particle theory and progressing to the appropriate quantum theory (i.e. moving from left to right in the upper row of the figure). It is also possible to describe quantum particles which interact with a classical field (e.g. a quantum charged particle in an electromagnetic field). However, it is perhaps not yet clear how to fill in the box in the lower-right-hand corner. That is, how to make a quantum theory of fields. We'll see how this works in September - for now let us go over the other boxes in figure 1, beginning with the classical column on the left-hand side.

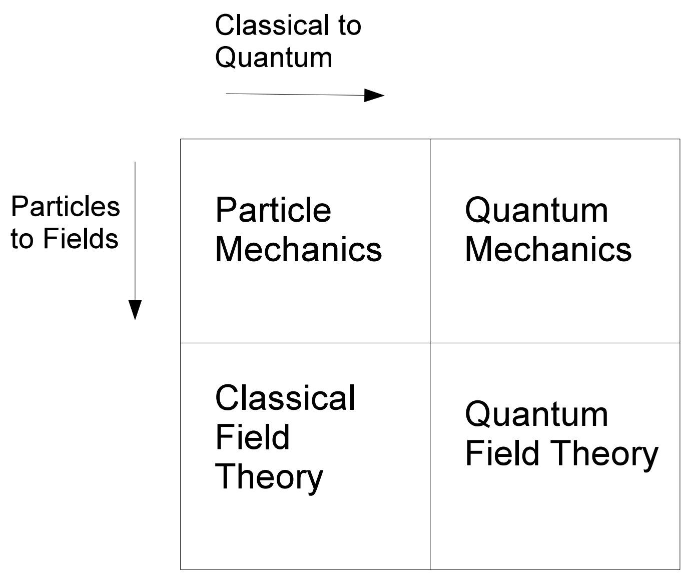

Figure 1: Schematic representation of the relationship between particles and fields, in both a classical and quantum setting. The QFT course concerns the lower right-hand corner.

1.1 Classical mechanics

1.1.1 Particles

In this section, we review the Lagrangian formalism for classical point particle mechanics. A given particle has a kinetic energy T, and a potential energy V. From these one may form the *Lagrangian*

$$L = T - V. \tag{1}$$

As an example, a particle moving in one dimension x with time coordinate t has

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{2}$$

where the dot denotes differentiation with respect to t. In general, V and T will depend on more than one space coordinate, and these may not necessarily be orthogonal. The coordinates may also have dimensions other than that of length (e.g. a dimensionless angle θ is the most convenient way to describe a pendulum).

Now consider a particle which follows a path between fixed positions x_1 and x_2, at times t_1 and t_2 respectively. There are infinitely many paths connecting these positions. However, a classical particle follows only one path, namely that which extremises the *action*

$$S = \int_{t_1}^{t_2} L(\dot{x}, x)dt. \tag{3}$$

Here we have continued to assume a one-dimensional system, but the generalisation to more complicated cases is straightforward. This is known as the *principle of least action*, as S is usually minimised, and we can use eq. (3) to find an equation for the path followed by the particle, as follows.

Let $\tilde{x}(t)$ be the path followed by the particle, defined above as the path which extremises the action. We may then perturb the path by a small amount, writing

$$x(t) = \tilde{x}(t) + \epsilon(t) \tag{4}$$

for some small correction $\epsilon(t)$, where $\epsilon(t_1) = \epsilon(t_2) = 0$ due to the fixed endpoints. This in turn will perturb the action, and substituting eq. (4) into eq. (3) one finds

$$\begin{aligned} S &= \textstyle\int_{t_1}^{t_2} L(\dot{\tilde{x}} + \dot{\epsilon}(t), \tilde{x}(t) + \epsilon(t))dt \\ &= \textstyle\int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{\epsilon}(t) + \left(\frac{\partial L}{\partial x}\right)\epsilon\right] dt + S[\tilde{x}(t)], \end{aligned} \tag{5}$$

where we have Taylor expanded L as a function of its arguments, and used the chain rule. One may perform an integration by parts in the first term to rewrite eq. (5) as

$$S = S[\tilde{x}(t)] + \delta S, \tag{6}$$

where

$$\delta S = \left[\epsilon(t)\left(\frac{\partial L}{\partial \dot{x}}\right)\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \left(\frac{\partial L}{\partial x}\right)\right]\epsilon(t)dt = 0, \tag{7}$$

where the final equality follows from the fact that the action is extremised by the path $\tilde{x}(t)$. In fact, the first term in eq. (7) vanishes due to the fact that $\epsilon(t) = 0$ at the endpoints, so that the principal of least action dictates that the second term vanishes, for arbitrary perturbations $\epsilon(t)$. Thus one has

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{8}$$

This is the *Euler-Lagrange equation*, and substituting eq. (2) we see that it is equivalent (in our one dimensional example) to

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x}, \tag{9}$$

which is Newton's Second Law. The Lagrangian method does a lot more, however, than allow us to rederive Newton's law. In particular:

1. Lagrangian methods greatly simplify the analyses of systems in many dimensions, and where the *generalised coordinates* may not be simple position variables (one may check that the Euler Lagrange equation has the same form for any generalised coordinate).

2. One may show quite generally in the Lagrangian approach that symmetries of a system (e.g. temporal or spatial translation invariance, rotational invariance) correspond to conserved quantities.

3. One may easily extend the Lagrangian approach to fields as well as particles, which we outline in the next section.

Before considering field theory, it is convenient to also introduce a few additional concepts. Firstly, given a Lagrangian L involving a coordinate x, one may define the *canonical momentum*

$$p = \frac{\partial L}{\partial \dot{x}}, \tag{10}$$

where p is said to be *conjugate* to x[1]. We may then define the *Hamiltonian*

$$H(p, x) = p\dot{x}(p, x) - L((\dot{x}(p, x), x), \tag{11}$$

where we have used the fact that $\dot{x}$ may be regarded as a function of x and p. Above, for example, $\dot{x} = p/m$ and one finds

$$H(x, p) = \frac{p^2}{2m} + V(x), \tag{12}$$

which is the total energy $(T+V)$ of the particle. When more than one coordinate is involved, the analogue of eq. (11) is

$$H = \sum_n p_n \dot{q}_n - L(q_n, p_n), \tag{13}$$

where the $\{q_n\}$ are the relevant generalised coordinates, and the canonical momenta are given by

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \tag{14}$$

One may again interpret H as the total energy of the system. Furthermore, one may replace the Euler-Lagrange equations with a more symmetric-looking set of first order equations. Differentiating H at constant p, one finds

$$\begin{aligned} \left(\frac{\partial H}{\partial x}\right)_p &= p\tfrac{\partial \dot{x}}{\partial x} - \tfrac{\partial L}{\partial \dot{x}}\tfrac{\partial \dot{x}}{\partial x} - \tfrac{\partial L}{\partial x} \\ &= \tfrac{\partial \dot{x}}{\partial x}\left(p - \tfrac{\partial L}{\partial \dot{x}}\right) - \tfrac{\partial L}{\partial x} \\ &= -\dot{p}, \end{aligned} \tag{15}$$

where we have used eq. (11) in the first line and eq. (8) in the second. Similarly, one may find

$$\left(\frac{\partial H}{\partial p}\right)_x = \dot{x}. \tag{16}$$

Equations (15) and (16) constitute *Hamilton's equations*, and they are particularly useful in relating symmetries to conservation laws. For example, if H does not depend on the position x, then one finds

[1]For our above example, $p = m\dot{x}$ and corresponds to the familiar definition of momentum. This is not necessarily the case for more complicated generalised coordinates.

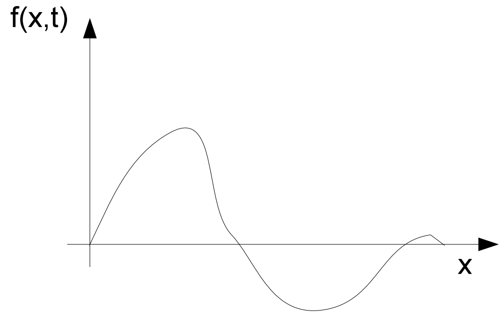

Figure 2: The transverse displacement $f(x,t)$ along a string.

that momentum is conserved directly from eq. (15). As another example, consider differentiating H with respect to t:

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} \\ &= \frac{\partial H}{\partial t}, \end{aligned} \tag{17}$$

where we have used Hamilton's equations in the second line. It follows that the left-hand side vanishes if H does not explicitly depend on t. That is, energy is conserved in a system which is invariant under time translations.

1.1.2 Fields

In classical mechanics, one does not only have particles. There may also be fields, which create forces on particles e.g. charged particles in an electromagnetic field, massive particles in a gravitational field. A particle is specified by its trajectory in space $x_i(t)$, where $\{x_i\}$ are the coordinates. A field, on the other hand, is specified by its value in space and time. For example, a given electric field may be written as $\boldsymbol{E}(\boldsymbol{x},t)$, denoting explicitly the dependence on all space and time coordinates. The quantity $\boldsymbol{E}$ here is a vector. However, fields in general may be scalar, vector or more generalised types of mathematical object such as spinors and tensors. We will consider only the case of scalar fields in the QFT course, although more complicated examples will occur in the QED / QCD course.

As a simple example of a scalar field, let us consider an example of a field in one space dimension, namely the transverse displacement $f(x,t)$ of a guitar string. This depends on the position along the string, x, as well as the time t thus is indeed a field. How do we describe such a system? Just as in the case of point particles, we may construct the Lagrangian $L = T - V$. The kinetic energy at time t is given by

$$T = \int dx \frac{1}{2}\rho \dot{f}^2(x,t). \tag{18}$$

Here ρ is the density (mass per unit length) of the string, and $\dot{f}(x,t)$ the transverse speed of the string at position x. One then integrates over all positions to obtain the total energy. Similarly, the total potential energy is

$$V = \int dx \frac{1}{2}\sigma (f')^2(x,t), \tag{19}$$

where σ is the tension of the string, and the prime denotes differentiation with respect to x rather than t. One may then write

$$L = \int dx \mathcal{L}(f, \dot{f}, f', t), \tag{20}$$

where

$$\mathcal{L} = \frac{1}{2}\rho\dot{f}^2 - \frac{1}{2}\sigma(f')^2, \tag{21}$$

is the *Lagrangian density*, which we may also write as $\mathcal{T} - \mathcal{V}$, with $\mathcal{T}$ and $\mathcal{V}$ the kinetic and potential energy densities respectively. This structure is analagous to the point particle case, but note that the field, and not the coordinate x, appears in the Lagrangian density. The principle of least action states that the action

$$S = \int dt \int dx \mathcal{L}(f, \dot{f}, f', t) \tag{22}$$

should be extremised. This leads to the field theory version of the Euler-Lagrange equation:

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t f)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x f)} = \frac{\partial \mathcal{L}}{\partial f}. \tag{23}$$

In the course, we will be concerned with applying the above ideas in a relativistic context, and the appropriate generalisation of the Euler-Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)} = \frac{\partial \mathcal{L}}{\partial f}, \tag{24}$$

where $\partial_\mu = \partial/\partial x^\mu$.

In the QFT course, we will give a fuller exposition of classical field theory. For now, let us note that, analagously to point particles, one may define the canonical momentum conjugate to the field f:

$$\pi(x, t) = \frac{\partial \mathcal{L}}{\partial \dot{f}}. \tag{25}$$

One may then construct the *Hamiltonian density*

$$\mathcal{H}(f, \pi) = \pi\dot{f} - \mathcal{L}, \tag{26}$$

whose integral over x gives the total energy carried by the field.

1.2 Quantum Mechanics

In the last section, we reviewed the Lagrangian treatment of classical mechanics, for point particles and for fields. In this section, we consider quantum mechanics. More specifically, we consider the quantum theory of point particles. We will not consider the quantum theory of fields here (i.e. that is the point of the course!). However, we will later see that some of the concepts introduced here are relevant to QFT.

To turn a classical point particle theory into a quantum theory, one replaces position and momentum variables by operators. To be more specific, let us take the one-dimensional theory whose Hamiltonian is given by eq. (12). Considering the position-space wavefunction $\Psi(x, t)$ for the particle, the position, momentum and energy operators are given by

$$\hat{x} = x, \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}, \tag{27}$$

from which it is easily shown that the position and momentum operators satisfy

$$[\hat{x}, \hat{p}] = i\hbar. \tag{28}$$

The wavefunction satisfies the Schrödinger equation

$$\hat{H}\Psi(x,t) = i\hbar\frac{\partial \Psi}{\partial t}, \tag{29}$$

where $\hat{H}$ is the Hamiltonian operator (obtained straightforwardly from eq. (12))

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{30}$$

Inserting eq. (27), one obtains the explicit differential equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x) = i\hbar\frac{\partial \Psi}{\partial t}. \tag{31}$$

In the next section, we consider a particular system which will later be useful for QFT, namely the quantum harmonic oscillator.

1.2.1 The Quantum Harmonic Oscillator

The harmonic oscillator is defined by the Hamiltonian of eq. (12) with

$$V(x) = \frac{1}{2}m\omega^2 x^2, \tag{32}$$

where m is the mass of the particle in the harmonic potential well, and ω is the period of oscillation. In the quantum theory, one may use the fact that $\hat{H}$ does not depend explicitly upon t to set $\Psi(x,t) = \psi(x)\Theta(t)$ in eq. (29). Dividing also by Ψ, one finds

$$\frac{1}{\psi(x)}\hat{H}\psi(x) = \frac{i\hbar}{\Theta(t)}\frac{d\Theta(t)}{dt}. \tag{33}$$

The left-hand and right-hand sides are functions only of x and t respectively, thus must be separately equal to a constant E. The left-hand side then gives

$$\hat{H}\psi(x) = E\psi(x), \tag{34}$$

which is the *time-independent Schrödinger equation*. Given that $\hat{H}$ represents the total energy operator of the system, it follows that solutions of eq. (34) are energy eigenfunctions with energy E. The right-hand side can be solved easily to give the time dependence of the energy eigenfunctions, and need not be considered in what follows.

One may show that, as expected for a confining potential well, there is a discrete spectrum of energy eigenvalues $\{E_n\}$. One way to show this is to explicitly solve eq. (31) as a differential equation with the appropriate form of $V(x)$. However, there is a simpler way to obtain the energy spectrum which will be useful later on when we consider QFT.

Let us define the operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{1}{m\omega\hbar}}\hat{p}\right), \tag{35}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{1}{m\omega\hbar}}\hat{p}\right). \tag{36}$$

(37)

It is straightforward to check from eq. (28) that

$$[a, a^\dagger] = 1. \tag{38}$$

Furthermore, the Hamiltonian operator may be written in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \hbar\omega, \tag{39}$$

from which it follows that

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{40}$$

Now consider the set of energy eigenstates $\{|n\rangle\}$. That is,

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{41}$$

Acting on the left with $\hat{a}^\dagger$ and using eq. (40), one finds

$$\hat{a}^\dagger \hat{H}|n\rangle = (\hat{H}\hat{a}^\dagger - \hbar\omega\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{42}$$

and thus

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (E_n + \hbar\omega)(\hat{a}^\dagger|n\rangle). \tag{43}$$

We therefore see that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E_{n+1} i.e.

$$\hat{a}^\dagger|n\rangle \propto |n+1\rangle. \tag{44}$$

Likewise, one may show that

$$\hat{H}(\hat{a}|n\rangle) = (E_n - \hbar\omega(\hat{a}|n\rangle). \tag{45}$$

and thus that

$$\hat{a}|n\rangle \propto |n-1\rangle. \tag{46}$$

Based on eqs. (44) and (46), $\hat{a}^\dagger$ and $\hat{a}$ are referred to as *raising* and *lowering* operators respectively, and collectively as *ladder operators*.

Let us now show that there is a minimum energy state. This follows from the fact that the norm of any state $|\phi\rangle$ must be non-negative i.e.

$$\langle\phi|\phi\rangle \geq 0. \tag{47}$$

Consider the state $|\phi\rangle = \hat{a}|n\rangle$. The conjugate of this state, as can be found from eqs.(35) and (36), is $\langle n|\hat{a}^\dagger$, so that

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle \geq 0. \tag{48}$$

However, using eq. (39) we may rewrite this as

$$\begin{aligned} \langle n| \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)|n\rangle \quad &= \langle n| \left(\tfrac{E_n}{\hbar\omega} - \tfrac{1}{2}\right)|n\rangle \\ &= \tfrac{E_n}{\hbar\omega} - \tfrac{1}{2} \geq 0, \end{aligned} \tag{49}$$

where we have assumed the normalisation $\langle n|n\rangle$ in the second line. We thus find that the minimum energy state $|0\rangle$ has energy $E_0 = \hbar\omega/2$. The first excited state of the system is

$$|1\rangle = \hat{a}^\dagger|0\rangle \tag{50}$$

and has energy $(1+1/2)\hbar\omega$. The n_{th} excited state has energy $(n+1/2)\hbar\omega$, and is given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle, \tag{51}$$

where the prefactor guarantees that $\langle n|n\rangle = 1$. Thus we may write the action of the Hamiltonian (eq. (39)) as

$$\hat{H}|n\rangle = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega|n\rangle = \left(\hat{n} + \frac{1}{2}\right)|n\rangle, \tag{52}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the *number operator*. That is, $|n\rangle$ is an eigenstate of $\hat{n}$, which has energy eigenvalue $(n+1/2)\hbar\omega$ so that the number n represents the number of quanta with energy $\hbar\omega$ which have been introduced to the ground state $|0\rangle$ of the system. We may think of $\hat{a}^\dagger$ and $\hat{a}$ as creating and annihilating quanta respectively, i.e. as *creation* and *annihilation* operators.

1.3 Problems

Here are some problems to check your understanding of the above material. Questions marked with an asterisk are optional. Solutions to all of the problems will be provided at the school.

Classical Mechanics

1. Show from the Euler-Lagrange equation that adding a constant to the potential energy V of a system is irrelevant i.e. does not affect the equations of motion.

2. A particle of mass m moves in the gravitational potential near the Earth's surface with height z. Write down the Lagrangian, and show that the equations of motion are what you would expect. Derive also the Hamiltonian, and show that this is indeed the total energy of the particle.

3. (*) Write down the Lagrangian for the motion of a particle of mass m in a spherically symmetric potential $V = V(r)$. You may use the fact that the dot product in spherical polar coordinates (r, θ, ϕ) is given by

$$\boldsymbol{a}\cdot\boldsymbol{b} = \begin{pmatrix} a_r & a_\theta & a_\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} b_r \\ b_\theta \\ b_\phi \end{pmatrix}.$$

   Show that the radial equation of motion is given by

$$\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{m}\frac{dV}{dr} = 0.$$

   Also show that the Hamiltonian of the particle is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r).$$

   What conserved quantity does the independence of the Hamiltonian on the azimuthal coordinate ϕ correspond to?

Classical Field Theory

4. Give examples of (a) a scalar field in three dimensions; (b) a vector field in three dimensions. Point particle mechanics can be considered as a field theory in zero space dimensions. Why?

5. For the field theory whose Lagrangian density is given by eq. (21), show that the Euler-Lagrange equation (eq. (24)) has the form
$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\rho}{\sigma}\frac{\partial^2 f(x,t)}{\partial t^2}.$$
Do you recognise this equation? Write down the canonical momentum conjugate to f, and the Hamiltonian density. Verify that the integral of the Hamiltonian density is the total energy carried by the field.

6. (*) Consider the action
$$S = \int d^4x\, \mathcal{L}(\partial_\mu\phi, \phi)$$
for a relativistic field ϕ. Show that the principle of least action implies the Euler-Lagrange equation of eq. (24). The proof is analagous to the point particle case.

Quantum Mechanics

7. One may derive the Schrödinger equation for a free particle ($V = 0$) in three space dimensions by starting with the kinetic energy definition
$$E = \frac{\boldsymbol{p}^2}{2m},$$
and substituting
$$E \to ih\frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i\nabla$$
(which operate on a wavefunction $\Psi(\boldsymbol{x}, t)$) to get
$$\left[-\frac{\hbar^2}{2m}\nabla^2\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$
Show that carrying out the same procedure on the relativistic kinetic energy relation
$$E^2 = c^2\boldsymbol{p}^2 + m^2c^4,$$
one gets the equation
$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\Phi = 0 \qquad (53)$$
for some wavefunction $\Phi(\boldsymbol{x}, t)$, where
$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$
is the d'Alembertian operator. Equation (53) is the *Klein-Gordon equation* for relativistic scalar particles.

8. Verify eqs. (38), (39) and (40).

9. (*) Check the normalisation of the state $|n\rangle$ in eq. (51). Hint: show that if $|n\rangle$ is correctly normalised, then so is $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$.

2 Introduction to QED and QCD

2.1 Rotations, Angular Momentum and the Pauli Matrices

1. Show that a 3-dimensional rotation can be represented by a 3×3 orthogonal matrix R with determinant $+1$ (Start with $\boldsymbol{x}' = R\,\boldsymbol{x}$, and impose $\boldsymbol{x}' \cdot \boldsymbol{x}' = \boldsymbol{x} \cdot \boldsymbol{x}$). Such rotations form the special orthogonal group, $SO(3)$.

2. For an infinitesimal rotation, write $R = \mathbb{1} + iA$ where $\mathbb{1}$ is the identity matrix and A is a matrix with infinitesimal entries. Show that A is antisymmetric (the i is there to make A hermitian).

3. Parameterise A as

$$A = \begin{pmatrix} 0 & -ia_3 & ia_2 \\ ia_3 & 0 & -ia_1 \\ -ia_2 & ia_1 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} a_i L_i \tag{54}$$

where the a_i are infinitesimal and verify that the L_i satisfy the angular momentum commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \tag{55}$$

Note that the Einstein summation convention is used here. Compute $L^2 \equiv L_1^2 + L_2^2 + L_3^2$. What is the interpretation of L^2 ?

4. The Pauli matrices σ_i are,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{56}$$

Verify that $\frac{1}{2}\sigma_i$ satisfy the same commutation relations as L_i.

5. Verify the following identity for the Pauli σ matrices:

$$\boldsymbol{\sigma} \cdot \mathbf{a}\; \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i\; \boldsymbol{\sigma} \cdot \mathbf{a} \wedge \mathbf{b} \;\;, \tag{57}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, $\mathbf{a} \cdot \mathbf{b}$ is their inner product, and $\mathbf{a} \wedge \mathbf{b}$ is their cross product.

2.2 Probability Density and Current Density

6. Starting from the Schrödinger equation for the wave function $\psi(\boldsymbol{x}, t)$, show that the probability density $\rho = \psi^* \psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \;\;, \tag{58}$$

where

$$\boldsymbol{J} = \frac{\hbar}{2im}\left[\psi^*(\boldsymbol{\nabla}\psi) - (\boldsymbol{\nabla}\psi^*)\psi\right] \;\; . \tag{59}$$

What is the interpretation of $\boldsymbol{J}$?

2.3 Scattering in Quantum Mechanics

7. The scattering of a particle of mass m by a potential $V(\boldsymbol{x})$ can be described in nonrelativistic quantum mechanics by using the Born approximation. The result for the scattering cross section in the first-order Born approximation is given by

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\,|\widetilde{V}(\boldsymbol{q})|^2 \ , \tag{60}$$

where $d\sigma/d\Omega$ is the differential cross section in the solid angle element, $\widetilde{V}(\boldsymbol{q})$ is the Fourier transform of the potential given by

$$\widetilde{V}(\boldsymbol{q}) = \int d^3x\ e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\ V(\boldsymbol{x}) \ , \tag{61}$$

and $\hbar\boldsymbol{q}$ is the momentum transferred in the scattering. Verify that for a spherically symmetric potential, $V(\boldsymbol{x}) = f(r)$ with $r = |\boldsymbol{x}|$, the integral in Eq. (61) takes the form

$$\int d^3\boldsymbol{x}\ e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\ V(\boldsymbol{x}) = \frac{4\pi}{|\boldsymbol{q}|}\int_0^\infty dr\ r\ \sin(|\boldsymbol{q}|r)\ f(r)\ . \tag{62}$$

8. Using Eqs. (60) and (62), show that the cross section for the scattering by a Yukawa potential of the form

$$V(\boldsymbol{x}) = C\ \frac{e^{-\mu|\boldsymbol{x}|}}{|\boldsymbol{x}|} \quad , \qquad \mu > 0 \ , \tag{63}$$

is given by

$$\frac{d\sigma}{d\Omega} = \frac{4C^2m^2}{\hbar^4(\mu^2+\boldsymbol{q}^2)^2} \ . \tag{64}$$

9. By making the replacements

$$C \to \alpha \quad , \qquad \mu \to 0 \tag{65}$$

in Eqs. (63) and (64), obtain the cross section for the scattering by a Coulomb potential. Expressing the result in terms of the incoming particle momentum $\boldsymbol{p} = \hbar\boldsymbol{k} = m\boldsymbol{v}$ and the scattering angle θ via the kinematic relation $\boldsymbol{q}^2 = 4\boldsymbol{k}^2\sin^2(\theta/2)$, show that the Coulomb scattering cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Coul.}} = \frac{\alpha^2}{4\ \boldsymbol{p}^2\ \boldsymbol{v}^2\ \sin^4(\theta/2)} \ . \tag{66}$$

This equals the classical Rutherford scattering cross section.

2.4 Four-vectors

10. A Lorentz transformation on the coordinates $x^\mu = (ct, \boldsymbol{x})$ can be represented by a 4×4 matrix Λ as follows:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{67}$$

For a boost along the x-axis to velocity v, show that

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{68}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$ as usual.

11. By imposing the condition

$$g_{\mu\nu}x'^{\mu}x'^{\nu} = g_{\mu\nu}x^{\mu}x^{\nu} \tag{69}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{70}$$

show that

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g \Lambda = g \tag{71}$$

This is the analogue of the orthogonality relation for rotations. Check that it works for the Λ given by equation (68) above.

12. Now introduce

$$x_{\mu} = g_{\mu\nu}x^{\nu} \tag{72}$$

and show, by reconsidering equation (69) using $x^{\mu}x_{\mu}$, or otherwise, that

$$x'_{\mu} = x_{\nu}(\Lambda^{-1})^{\nu}{}_{\mu} \tag{73}$$

13. Vectors A^{μ} and B_{μ} that transform like x^{μ} and x_{μ} are sometimes called *contravariant* and *covariant* respectively. A simpler pair of names is *vector* and *covector*. A particularly important covector is obtained by letting $\partial/\partial x^{\mu}$ act on a scalar ϕ:

$$\frac{\partial \phi}{\partial x^{\mu}} \equiv \partial_{\mu}\phi \tag{74}$$

Show that ∂_{μ} does transform like x_{μ} and not x^{μ}.

14. The dot product of two 4-vectors x and y is defined as

$$x \cdot y \equiv x^{\mu}y^{\nu}g_{\mu\nu} = x^{\mu}y_{\mu} = x_{\mu}y^{\mu} \quad . \tag{75}$$

In particular x^2 is given by

$$x^2 = x \cdot x = x^{\mu}x_{\mu} \quad . \tag{76}$$

If $x \cdot y = 0$, x and y are orthogonal. A 4-vector x is said to be light-like if $x^2 = 0$; time-like if $x^2 > 0$; space-like if $x^2 < 0$.
Show that any 4-vector orthogonal to a time-like 4-vector is space-like.

2.5 Relativistic Kinematics

15. The four-velocity is defined as

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \quad , \tag{77}$$

where $dx^{\mu} = (c\ dt,\ d\boldsymbol{x})$, and $d\tau$ is the proper time given by

$$d\tau = \frac{1}{c}\sqrt{dx^{\mu}dx_{\mu}} \quad . \tag{78}$$

Show that

$$u^{\mu} = \left(\frac{c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\, , \ \frac{\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}} \right) \quad , \quad \text{where } \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \quad , \tag{79}$$

and that

$$u^2 = u^{\mu}u_{\mu} = c^2 \quad . \tag{80}$$

16. The four-momentum of a particle of mass m is defined as

$$p^\mu = m u^\mu = \left(\frac{E}{c} \ , \ \boldsymbol{p} \right) \tag{81}$$

where E is the energy and $\boldsymbol{p}$ is the three-momentum. Show that

$$p^2 = m^2 c^2 \tag{82}$$

and that

$$E^2 = c^2 \boldsymbol{p}^2 + m^2 c^4 \ . \tag{83}$$

17. The kinetic energy of a particle of momentum $p^\mu = (E/c, \boldsymbol{p})$ and mass m is given by

$$T = E - mc^2 \ . \tag{84}$$

Show that in the non-relativistic limit $|\boldsymbol{p}| \ll mc$ this gives

$$T = \frac{\boldsymbol{p}^2}{2m} \ \left[1 + \mathcal{O}\left(\frac{\boldsymbol{p}^2}{m^2 c^2} \right) \right] \ . \tag{85}$$

18. A particle of mass M comes to rest and decays into a particle of mass m and a massless particle. Show that the kinetic energy of the particle of mass m is

$$T = \frac{(M-m)^2 \ c^2}{2M} \ . \tag{86}$$

19. A particle of mass M decays into two massless particles with energies E_1 and E_2. Determine an expression for the angle θ between the two final particles. Show that the minimum possible value of this angle obeys the relation

$$\sin(\theta_{min}/2) = \sqrt{1 - \boldsymbol{v}^2/c^2} \tag{87}$$

where $\boldsymbol{v}$ is the velocity of the decaying particle.

2.6 Electromagnetism

20. Classical electromagnetism is summed up in Maxwell equations, two homogeneous,

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \quad , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad , \tag{88}$$

and two inhomogeneous,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \quad , \qquad \boldsymbol{\nabla} \wedge \boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{J} \quad . \tag{89}$$

The homogeneous ones are automatically solved by introducing the potentials ϕ and $\mathbf{A}$ as follows

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}} \quad , \qquad \boldsymbol{B} = \boldsymbol{\nabla} \wedge \boldsymbol{A} \quad . \tag{90}$$

Show that the pair of inhomogeneous Maxwell equations in terms of the potentials take the form

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \right) \phi - \frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{A} \right) &= \rho \quad , \\ \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \right) \boldsymbol{A} + \boldsymbol{\nabla} \left(\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{A} \right) &= \boldsymbol{J} \quad . \end{aligned} \tag{91}$$

21. Show that Maxwell equations are invariant under the gauge transformations

$$\phi \to \phi + \dot{\lambda} \ , \quad \boldsymbol{A} \to \boldsymbol{A} - \boldsymbol{\nabla}\lambda \quad , \tag{92}$$

where λ is an arbitrary function.

22. Define the four-potential

$$A^\mu = (\phi \ , \ \boldsymbol{A}) \tag{93}$$

and the four-current

$$j^\mu = (\rho \ , \ \boldsymbol{J}) \quad . \tag{94}$$

Verify that the equations of motion (91) can be rewritten in relativistically covariant form as

$$\partial^2 A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu \quad , \tag{95}$$

where $\partial^\mu = (\partial/\partial t, -\boldsymbol{\nabla})$, and the gauge transformations (92) can be rewritten as

$$A^\mu \to A^\mu + \partial^\mu \lambda \quad . \tag{96}$$

23. Verify that the formula

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{97}$$

expresses the fields in terms of the potentials according to Eq. (90), and that the inhomogeneous Maxwell equations (89) can be written in relativistically covariant form as

$$\partial_\mu F^{\mu\nu} = j^\nu \ . \tag{98}$$

24. The Lorentz gauge-fixing condition is defined as

$$\partial_\nu A^\nu = 0 \ . \tag{99}$$

Eq. (99) fixes the gauge up to residual gauge transformations of the form (96) with

$$\partial^2 \lambda = 0 \ . \tag{100}$$

In the Lorentz gauge the equations of motion (95) take the form

$$\partial^2 A^\mu = j^\mu \ . \tag{101}$$

By Fourier transformation on Eq. (101) show that

$$\widetilde{A}^\mu = \frac{-g^{\mu\nu}}{q^2} \, \widetilde{j}^\mu \quad , \tag{102}$$

where

$$\widetilde{A}^\mu = \int d^4x \ e^{-iqx} A^\mu \ , \quad \widetilde{j}^\mu = \int d^4x \ e^{-iqx} j^\mu \ . \tag{103}$$